

Information-Preserving Markov Aggregation

Bernhard C. Geiger*, Christoph Temmel†

*Signal Processing and Speech Communication Laboratory, Graz University of Technology, Austria

†Department of Mathematics, VU University - Faculty of Sciences, Amsterdam, The Netherlands
geiger@ieee.org, ctc@temmel.me

Abstract—We present a sufficient condition for a non-injective function of a Markov chain to be a second-order Markov chain with the same entropy rate as the original chain. This permits an information-preserving state space reduction by merging states or, equivalently, lossless compression of a Markov source on a sample-by-sample basis. The cardinality of the reduced state space is bounded from below by the node degrees of the transition graph associated with the original Markov chain.

We also present an algorithm listing all possible information-preserving state space reductions, for a given transition graph. We illustrate our results by applying the algorithm to a bi-gram letter model of an English text.

Index Terms—lossless compression, Markov chain, model order reduction, n -gram model

I. INTRODUCTION

Markov chains are ubiquitously used in many scientific fields, ranging from machine learning and systems biology over speech processing to information theory, where they act as models for sources and channels. In some of these fields, however, the state space of the Markov chain is too large to allow either proper training of the model (see n -grams in speech processing [1]) or its simulation (as in chemical reaction networks [2]).

One way to reduce the cardinality of the state space of a Markov chain is to merge states, which is equivalent to feeding the process through a non-injective function. The merging usually depends on the cost function; candidate methods either rely on the Fiedler vector or other spectral criteria [3], [4], or on the Kullback-Leibler divergence rate w.r.t. some reference process [5], [6].

In addition to the model information lost by merging, the obtained process does, in general, not possess the Markov property. Modeling it as a Markov chain on the reduced state space as suggested in, e.g., [5], typically leads to an additional loss of model information. The same holds for Markov models obtained from clustered training data, as, e.g., for the n -gram class model in [7]. Consequently, there is a trade-off between cardinality of the state space, model complexity, and information loss.

Recently, we have shown the existence of sufficient conditions on a Markov chain and a non-injective function merging its states such that the obtained process is not only a k th-order Markov chain (which is desirable from a computational point-of-view), but also preserves full model information [8]. While the former property is commonly referred to as *lumpability*, the latter is a rather surprising one: whereas, in principle, stationary sources can be compressed efficiently by assigning

codewords to *blocks* of samples, our result shows that in some cases lossless compression is possible on a sample-by-sample basis.

Extending our previous results, we show in Section III, using spectral theory of graphs, that for an information-preserving compression, the number of input sequences merged to the same output sequence is bounded independently of the sequence length. This result allows us to estimate the minimum cardinality of the reduced state space based on the degree structure of the transition graph of the original Markov chain. Furthermore, we prove that, if a specific partition of the original state space satisfies the sufficient conditions for k th-order Markovity and information-preservation, then so does every refinement of this partition. Section IV focuses on second-order Markov chains, due to their computationally desirable properties, and presents an iterative algorithm listing all possible partitions satisfying the abovementioned sufficient conditions. To illustrate the algorithm, we introduce a simple toy example in Section V before analyzing a bi-gram letter model in Section VI.

II. PRELIMINARIES & NOTATION

Throughout this work, we deal with an irreducible, aperiodic, homogeneous Markov chain \mathbf{X} on a finite state space \mathcal{X} and with transition matrix \mathbf{P} . Let X_n be the n th sample of the process, and let $X_i^j := \{X_i, X_{i+1}, \dots, X_j\}$. We assume that \mathbf{X} is stationary, i.e., that the initial distribution of the chain coincides with its invariant distribution $\boldsymbol{\mu}$. Hence, for every n , the distribution P_{X_n} of X_n equals $\boldsymbol{\mu}$.

We consider a surjective *lumping function* $g: \mathcal{X} \rightarrow \mathcal{Y}$, with $\text{card}(\mathcal{X}) =: N > M := \text{card}(\mathcal{Y}) \geq 2$. Abusing notation, we extend g to $\mathcal{X}^n \rightarrow \mathcal{Y}^n$ coordinate-wise and denote by $g^{-1}[y]$ the preimage of y under g . We call the stationary stochastic process \mathbf{Y} , defined by $Y_n := g(X_n)$, the *lumped process* and the tuple (\mathbf{P}, g) the *lumping*.

Since the lumping function is non-injective, a loss of information may occur, which we quantify by the conditional entropy rate

$$\bar{H}(\mathbf{X}|\mathbf{Y}) := \lim_{n \rightarrow \infty} \frac{1}{n} H(X_1^n | Y_1^n) = \bar{H}(\mathbf{X}) - \bar{H}(\mathbf{Y}) \quad (1)$$

where $H(\cdot)$ and $\bar{H}(\cdot)$ denote the entropy and the entropy rate (if it exists) of the argument, respectively. The lumping (\mathbf{P}, g) is *information-preserving* iff $\bar{H}(\mathbf{X}|\mathbf{Y}) = 0$.

III. PREVIOUS RESULTS & EXTENSIONS

We summarize several definitions and results from [8] relevant to this work:

Definition 1 (Preimage Count). The *preimage count of length n* is the random variable

$$T_n := \sum_{\mathbf{x} \in g^{-1}[Y_1^n]} [\Pr(X_1^n = \mathbf{x}) > 0] \quad (2)$$

where $[A] = 1$ if A is true and zero otherwise (Iverson bracket).

In other words, the preimage count maps each sequence of length n of the output process \mathbf{Y} to the cardinality of the realizable portion of its preimage.

The following characterization holds [8, Thm. 1]:

$$\bar{H}(\mathbf{X}|\mathbf{Y}) = 0 \Leftrightarrow \exists C < \infty: \Pr(\sup_{n \rightarrow \infty} T_n \leq C) = 1 \quad (3a)$$

$$\bar{H}(\mathbf{X}|\mathbf{Y}) > 0 \Leftrightarrow \exists C > 1: \Pr(\liminf_{n \rightarrow \infty} \sqrt[n]{T_n} \geq C) = 1 \quad (3b)$$

i.e., that an almost-surely bounded preimage count (for arbitrary sequence length n) is equivalent to a vanishing information loss rate.

The information-preserving case (3a) can be strengthened to a deterministic version:

Proposition 1 (Bounded Preimage Count).

$$\bar{H}(\mathbf{X}|\mathbf{Y}) = 0 \Leftrightarrow \exists C < \infty: \sup_{n \rightarrow \infty} T_n \leq C. \quad (4)$$

Proof: See Appendix. ■

An interesting line for future research would be to show a deterministic analog of (3b) and its direct derivation from the Shannon-McMillan-Breiman theorem [9, Ch. 16.8].

As a corollary to Proposition 1 we get

Corollary 1. *An information-preserving lumping (\mathbf{P}, g) satisfies*

$$M \geq \min_i d_i \quad (5)$$

where $d_i := \sum_{j=1}^N [P_{i,j} > 0]$ is the out-degree of state i .

Proof: See Appendix. ■

Corollary 1 upper-bounds the possible state space reduction of an information-preserving lumping. In particular, a Markov chain with a positive transition matrix \mathbf{P} does not admit an information-preserving lumping [8, Cor. 4]: In this case, all states have out-degree N , and the bound $M \geq N$ only holds for the trivial lumping.

Complementing this necessary condition for preservation of information, in [8, Prop. 10] we also gave a sufficient condition, additionally implying that \mathbf{Y} is a k th-order Markov chain, i.e., that $\forall n: \forall x \in \mathcal{X}, x_1^n \in \mathcal{X}^n:$

$$\Pr(X_n = x | X_1^{n-1} = x_1^n) = \Pr(X_n = x | X_{n-k}^{n-1} = x_{n-k}^{n-1}). \quad (6)$$

To this end, we introduced

Definition 2 (Single Forward Sequence [8, Def. 9]). For $k \geq 2$ a lumping (\mathbf{P}, g) has the *single forward k -sequence* property (short: SFS(k)) iff

$$\begin{aligned} \forall \mathbf{y} \in \mathcal{Y}^{k-1}, y \in \mathcal{Y}: \exists! \mathbf{x}' \in g^{-1}[\mathbf{y}]: \\ \forall x \in g^{-1}[y], \mathbf{x} \in g^{-1}[\mathbf{y}] \setminus \{\mathbf{x}'\}: \\ \Pr(X_2^k = \mathbf{x} | Y_2^k = \mathbf{y}, X_1 = x) = 0. \end{aligned} \quad (7)$$

Thus, for every realization of Y_1^n , the realizable preimage of Y_2^n is a singleton. Therefore, SFS(k) implies not only that \mathbf{Y} is k th-order Markov, but also that the lumping is information-preserving¹ [8, Prop. 10]. Note that SFS(k) is a property of the combinatorial structure of the transition matrix \mathbf{P} , i.e., it only depends on the location of its non-zero entries.

The SFS(k)-property has practical significance: Besides preserving, if possible, the information of the original model, those lumpings which possess the Markov property of any order are preferable from a computational perspective. Moreover, the corresponding conditions for the more desirable first-order Markov output, not necessarily information-preserving, are too restrictive in most scenarios (cf. [10, Sec. 6.3]).

The next result investigates a cascade of lumping functions. Let $g := h \circ f$, where $h: \mathcal{X} \rightarrow \mathcal{Z}$ and $f: \mathcal{Z} \rightarrow \mathcal{Y}$. We identify a function with the partition it induces on \mathcal{X} ; thus, abusing notation, we say that h is a *refinement* of g iff this holds for their induced partitions.

Proposition 2 (SFS(k) & Refinements). *If a lumping (\mathbf{P}, g) is SFS(k), then so is (\mathbf{P}, h) , for all refinements h of g .*

Proof: See Appendix. ■

Clearly, a refinement does not increase the loss of information, so information-preservation is preserved under refinements. In contrast, a refinement of a lumping yielding a k th-order Markov process \mathbf{Y} need not possess that property; the lumping to a single state has the Markov property, while a refinement of it generally has not.

All SFS(k)-lumpings lie within the intersection of information-preserving lumpings and lumpings yielding a k th-order Markov chain. However, the SFS(k)-property does not exhaust this intersection [8, Fig. 2]. In [8] we presented sufficient conditions for a lumping to be either information-preserving or to yield a k th-order Markov chain, respectively. We currently do not know if SFS(k) is identical to the intersection of these two sufficient conditions..

IV. AN ALGORITHM FOR SFS(2)-LUMPINGS

In this section, we present an algorithm listing all SFS(2)-lumpings, i.e., lumpings (\mathbf{P}, g) yielding a second-order Markov chain and preserving full model information.

SFS(2)-lumpings have the property that, for all $y_1, y_2 \in \mathcal{Y}$, from within a set $g^{-1}[y_1]$ at most one element in the set

¹Actually, SFS(k) implies more than $\bar{H}(\mathbf{X}|\mathbf{Y}) = 0$: It implies that a sequence of states of the reduced model uniquely determines the corresponding sequence of the original model, except for the first sample. Thus, the reduced model is in some sense “invertible”.

$g^{-1}[y_2]$ is accessible:

$$\begin{aligned} \forall y_1, y_2 \in \mathcal{Y}: \exists! x'_2 \in g^{-1}[y_2]: \\ \forall x_1 \in g^{-1}[y_1], x_2 \in g^{-1}[y_2] \setminus \{x'_2\}: P_{x_1, x_2} = 0. \end{aligned} \quad (8)$$

This gives rise to

Proposition 3. *An SFS(2)-lumping satisfies*

$$M \geq \max_i d_i. \quad (9)$$

Proof: We evaluate the rows of \mathbf{P} separately. All states x_2 accessible from state x_1 are characterized by $P_{x_1, x_2} > 0$. Any two states accessible from x_1 cannot be merged, since this would contradict (8). Thus, all states accessible from x_1 must have different images, implying $M \geq d_{x_1}$. The result follows by considering all states x_1 . ■

In particular, Proposition 3 implies that a transition matrix with at least one positive row does not admit an SFS(2)-lumping.

An algorithm listing all SFS(2)-lumpings, or SFS(2)-partitions, for a given transition matrix \mathbf{P} has to check the SFS(2)-property for all partitions of \mathcal{X} into at least $\max_i d_i$ non-empty sets. The number of these partitions can be calculated from the Stirling numbers of the second kind [11, Thm. 8.2.5] and is typically too large to allow an exhaustive search. Therefore, we use Proposition 2 to reduce the search space.

Starting from the trivial partition with N elements, we evaluate all possible merges of two states, i.e., all possible partitions with $N - 1$ sets, of which there exist $\frac{N(N-1)}{2}$. Out of these, we drop those from the list which do not possess the SFS(2)-property. The remaining set of *admissible pairs* is a central element of the algorithm.

We proceed iteratively: To generate all candidate partitions with $N - i$ sets, we perform all admissible pair-wise merges on all SFS(2)-partitions with $N - i + 1$ sets. An admissible pair-wise merge is a merge of two sets of a partition, where either set contains one element of the admissible pair. From the resulting partitions one drops those violating SFS(2) before performing the next iteration. Since this algorithm generates some partitions multiple times (see the toy example in Section V), in every iteration all duplicates are removed. The algorithm is presented in Table I.

Iterative generation of the partitions by admissible pair-wise merges allows application of Proposition 2, which reduces the number of partitions to be searched. If the number of admissible pairs is small compared to $\frac{N(N-1)}{2}$, then this reduction is significant. Inefficiencies in our algorithm caused by multiple considerations of the same partitions could be alleviated by adapting the classical algorithms for the partition generating problem [12], [13].

The actual choice of one of the obtained SFS(2)-partitions for model order reduction requires additional model-specific considerations: A possible criterion could be maximum compression (i.e., smallest entropy of the marginal distribution).

TABLE I
ALGORITHM FOR LISTING ALL SFS(2)-LUMPINGS

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1: procedure LISTLUMPINGS( $\mathbf{P}$ )
2:    $\text{admPairs} \leftarrow \text{GETADMISSIBLEPAIRS}(\mathbf{P})$ 
3:    $\text{Lumpings}(1) \leftarrow \text{merge}(\text{admPairs})$   $\triangleright$  Convert pairs to functions
4:    $n \leftarrow 1$ 
5:   while notEmpty( $\text{Lumpings}(n)$ ) do
6:      $n \leftarrow n + 1$ 
7:      $\text{Lumpings}(n) \leftarrow []$ 
8:     for  $h \in \text{Lumpings}(n-1)$  do
9:       for  $\{i_1, i_2\} \in \text{admPairs}$  do
10:         $g \leftarrow h$ 
11:         $g(h^{-1}(h(i_2))) \leftarrow g(i_1)$   $\triangleright i_1$  and  $i_2$  have same image.
12:        if  $g$  is SFS(2) then
13:           $\text{Lumpings}(n) \leftarrow [\text{Lumpings}(n); g]$ 
14:        end if
15:      end for
16:    end for
17:    Remove duplicates from  $\text{Lumpings}$ 
18:  end while
19:  return  $\text{Lumpings}$ 
20: end procedure


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21: function GETADMISSIBLEPAIRS( $\mathbf{P}$ )
22:    $\text{Pairs} \leftarrow []$ 
23:    $N \leftarrow \text{dim}(\mathbf{P})$ 
24:   for  $i_1 = 1 : N - 1$  do
25:     for  $i_2 = i_1 + 1 : N$  do
26:        $f \leftarrow \text{merge}(i_1, i_2)$   $\triangleright f$  merges  $i_1$  and  $i_2$ 
27:       if  $f$  is SFS(2) then
28:          $\text{Pairs} \leftarrow [\text{Pairs}; \{i_1, i_2\}]$ 
29:       end if
30:     end for
31:   end for
32:   return  $\text{Pairs}$ 
33: end function

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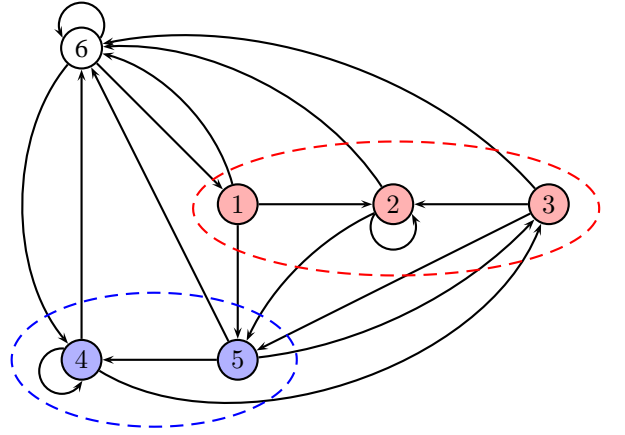


Fig. 1. A transition graph on 6 vertices with a lumping given by the partition $\{\{1, 2, 3\}, \{4, 5\}, \{6\}\}$. The lumping is of type SFS(2).

V. A TOY EXAMPLE

We illustrate our algorithm at the hand of a small example. Consider the six-state Markov chain with transition graph

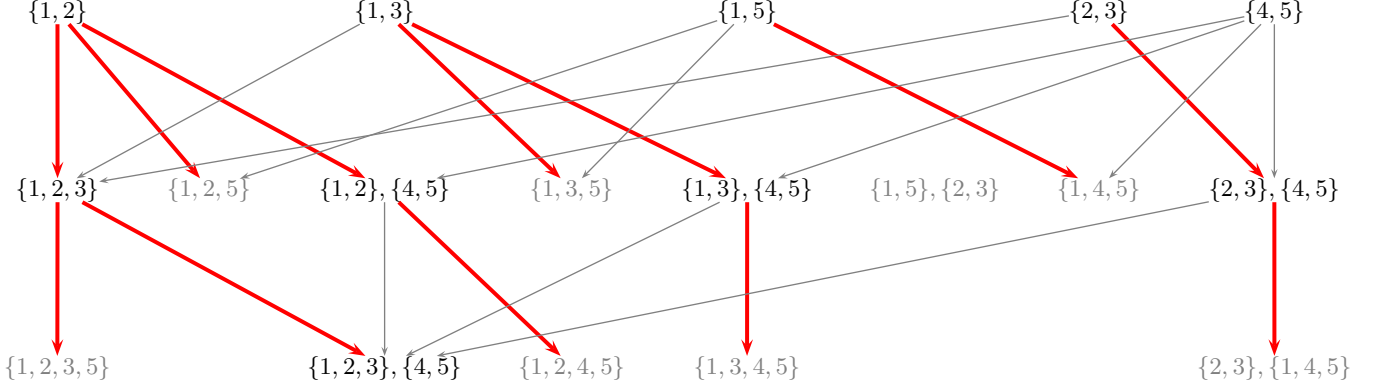


Fig. 2. An illustration of the algorithm of Table I at the hand of the example depicted in Fig. 1. The first row shows all admissible pairs, the algorithm runs through all rows (top to bottom) by merging according to the admissible pairs (left to right). Bold, red arrows indicate newly generated partitions, gray arrows indicate that this partition was already found and is thus removed as a duplicate. Gray partitions violate the SFS(2)-property. This figure lists all SFS(2)-partitions of \mathcal{X} (cf. Table II).

depicted in Fig 1, whose adjacency matrix \mathbf{A} is

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}. \quad (10)$$

Since all states have out-degree $d_i = 3$, lumpings to at least $M = 3$ states are considered. The lumping in Fig. 1 satisfies the SFS(2)-property. Fig. 2 shows the derivation of the lumping of Fig. 1 by our algorithm.

During initialization we evaluate all 15 possible pair-wise merges. Of these, we exclude all pairs where both members are accessible from the same state, i.e., $\{2, 5\}$, $\{2, 6\}$, $\{5, 6\}$, $\{3, 4\}$, $\{3, 6\}$, $\{4, 6\}$, $\{1, 4\}$, and $\{1, 6\}$. Furthermore, $\{2, 4\}$ and $\{3, 5\}$ are excluded too; the former because both states have self-loops, the latter because both states are connected in either direction. Only five pairs are admissible.

One admissible pair is $\{1, 2\}$, i.e., the function h merging $\{1, 2\}$ and, thus, inducing the partition $\mathcal{Z}_5 = \{\{1, 2\}, \{3\}, \{4\}, \{5\}, \{6\}\}$, satisfies SFS(2). With this h we enter the algorithm in the innermost loop (Table I, line 9). The algorithm performs pair-wise merges according to the five admissible pairs and obtains the following merges: $\{1, 2\}$, $\{1, 2, 3\}$, $\{1, 2, 5\}$, $\{\{1, 2\}, \{4, 5\}\}$; the first is a (trivial) duplicate (by performing a pair-wise merge according to $\{1, 2\}$) and the second is obtained twice (by pairing $\{1, 2\}$ with $\{1, 3\}$ and $\{2, 3\}$). Only $\{1, 2, 5\}$ violates SFS(2). The functions merging $\{1, 2, 3\}$ and $\{\{1, 2\}, \{4, 5\}\}$ are added to the list of lumping functions to four states, and the procedure is repeated for a different admissible pair.

For the next iteration, fix h such that it induces the partition $\mathcal{Z}_4 = \{\{1, 2, 3\}, \{4\}, \{5\}, \{6\}\}$. The five admissible pairs yield the non-trivial merges $\{1, 2, 3\}$, a duplicate which is obtained three times, $\{1, 2, 3, 5\}$, which violates SFS(2), and $\{\{1, 2, 3\}, \{4, 5\}\}$, which is the solution depicted in Fig. 1.

TABLE II
LIST OF SFS(2)-LUMPINGS OF THE EXAMPLE FOUND BY THE ALGORITHM

M	Partition \mathcal{Z}_M
6	$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}$
5	$\{1, 2\}, \{3\}, \{4\}, \{5\}, \{6\}$ $\{1, 3\}, \{2\}, \{4\}, \{5\}, \{6\}$ $\{1, 5\}, \{2\}, \{3\}, \{4\}, \{6\}$ $\{1\}, \{2, 3\}, \{3\}, \{4\}, \{6\}$ $\{1\}, \{2\}, \{3\}, \{4, 5\}, \{6\}$
4	$\{1, 2, 3\}, \{4\}, \{5\}, \{6\}$ $\{1, 2\}, \{3\}, \{4, 5\}, \{6\}$ $\{1, 3\}, \{2\}, \{4, 5\}, \{6\}$ $\{1\}, \{2, 3\}, \{4, 5\}, \{6\}$
3	$\{1, 2, 3\}, \{4, 5\}, \{6\}$

The algorithm terminates now, since every pair-wise merge of $\mathcal{Z}_3 = \mathcal{V} = \{\{1, 2, 3\}, \{4, 5\}, \{6\}\}$ either violates SFS(2) or is a duplicate. The list of all SFS(2)-lumpings found by the algorithm is given in Table II.

VI. CLUSTERING A BI-GRAM MODEL

We apply our algorithm to a bi-gram² letter model. Commonly used in speech processing [1, Ch. 6], n -grams (of which bi-grams are a special case) are $(n-1)$ th-order Markov models for the occurrence of letters or words. From a set of training data the relative frequency of the (co-)occurrence of letters or words is determined, yielding the maximum likelihood estimate of their (conditional) probabilities. In practice, for large n , even large training data cannot contain all possible sequences, so the n -gram model will contain a considerable amount of zero transition probabilities. Since this would lead to problems in, e.g., a speech recognition system, those entries are increased by a small constant to *smooth* the model, for example using Laplace's law [1, pp. 202].

Since by Proposition 3 an information-preserving lumping is more efficient for a sparse transition matrix, we refrain from smoothing and use the maximum likelihood estimates of

²Shannon used bi-grams, or *digrams* as he called them, as a second-order approximation of the English language [14].

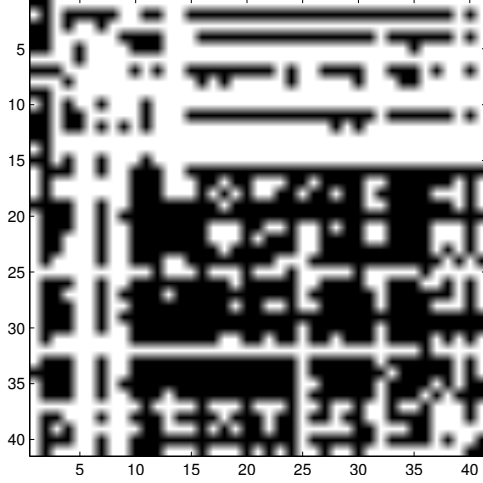


Fig. 3. The adjacency matrix of the bi-gram model of “The Great Gatsby”. The first two states are line break (LB) and space (‘ ’), followed by punctuations. The block in the lower right corner indicates interactions of letters and punctuation following letters.

the model parameters instead. We trained a Markov bi-gram letter model of F. Scott Fitzgerald’s “*The Great Gatsby*”, a text containing roughly 270000 letters. To reduce the alphabet size and, thus, the run-time of the algorithm, we replaced all numbers by ‘#’ and all upper case by lower case letters. We left punctuations unchanged, yielding a total alphabet size of 41. The adjacency matrix of the bi-gram model can be seen in Fig. 3; the maximum out-degree of the Markov chain is 37.

Of the 820 possible merges only 21 are admissible. Furthermore, there are 129, 246, and 90 SFS(2)-lumpings to sets of cardinalities 39, 38, and 37, respectively. There are only two admissible triples, namely $\{\text{LB}, ‘\$’, ‘x’\}$ and $\{\text{LB}, ‘(’, ‘x’\}$, where LB denotes the line break. Of the more notable pair-wise merges we mention $\{‘(’, ‘)’\}$, $\{‘(’, ‘z’\}$, and the merges of ‘#’ with colon, semicolon, and exclamation mark. Especially the first is intuitive, since parentheses can be exchanged to, e.g., ‘|’ while preserving the meaning of the symbol³.

We finally determined the lumping yielding maximum compression, i.e., the one for which $H(Y)$ is minimized. This lumping, merging $\{\text{LB}, ‘\$’, ‘x’\}$, $\{‘!’, ‘#’\}$, and $\{‘(’, ‘,’\}$, decreases the entropy from $H(X) = 4.3100$ to $H(Y) = 4.3044$. The entropies roughly correspond to the 4.03 bits derived for Shannon’s first-order model, which contains only 27 symbols [9, p. 170].

While the compression obtained with our algorithm seems negligible, we are confident that it improves for lumping n -grams with $n > 2$, since these appear to be even more sparse than the bi-gram model. For example, the transition chain of a tri-gram model trained with Fitzgerald’s text has a maximum out-degree of 32, compared to 37 for the bi-gram model. What

³Whether the symbol initiates or terminates a parenthetic expression is determined by whether the symbol is preceded or succeeded by a blank space. Unless parenthetic expressions are nested, simple counting distinguishes between initiation and termination.

currently prevents testing these claims is the lack of an efficient implementation of our algorithm.

Some merges which preserve information are not found by the algorithm: For example, since in the text every ‘q’ is followed by a ‘u’, and since no two ‘u’ occur in a row, it would be possible to merge these two letters: No information is lost, since if the merged state is visited once and left immediately, only a ‘u’ is possible. Conversely, if the merged state occurs twice it can only be an occurrence of ‘qu’. However, the model obtained by this merge does not possess the Markov property of any order, and thus violates SFS(2).

VII. CONCLUSION

We presented a sufficient condition for merging states of a Markov chain such that the resulting process is second-order Markov and has full model information. We furthermore developed an iterative algorithm finding all such merges for a given transition matrix. Finally, we presented a lower bound on the cardinality of the reduced state space depending on the maximum out-degree of the associated transition graph.

The application of our algorithm to a bi-gram letter model suggests its practical relevance for model-order reduction, e.g., for n -grams with $n > 2$. Future work shall investigate possible improvements of the algorithm as well as its complexity analysis.

ACKNOWLEDGMENTS

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APPENDIX

A. Proof of Proposition 1

We recall from [8] that $\bar{H}(\mathbf{X}|\mathbf{Y}) = 0$ implies, for all n ,

$$\begin{aligned} \forall \tilde{x}, \hat{x} \in \mathcal{X}, \mathbf{y} \in \mathcal{Y}^{n-2}: \\ \Pr(X_1 = \tilde{x}, Y_2^{n-1} = \mathbf{y}, X_n = \hat{x}) > 0 \\ \Rightarrow \exists! \mathbf{x} \in \mathcal{X}^{n-2}: \\ \Pr(X_2^{n-1} = \mathbf{x} | X_1 = \tilde{x}, Y_2^{n-1} = \mathbf{y}, X_n = \hat{x}) = 1. \end{aligned} \quad (11)$$

We thus obtain a bound on a realization t_n of the preimage count (i.e., for $Y_1^n = \mathbf{y}$)

$$\begin{aligned} t_n &= \sum_{\mathbf{x} \in g^{-1}[\mathbf{y}]} [\Pr(X_1^n = \mathbf{x}) > 0] \\ &= \sum_{\mathbf{x} \in \mathcal{X}^n} [\Pr(X_1^n = \mathbf{x} | Y_1^n = \mathbf{y}) > 0] \\ &= \sum_{\mathbf{x} \in \mathcal{X}^n} [\Pr(X_1 = x_1, X_n = x_n | Y_1^n = \mathbf{y}) > 0] \\ &\quad \times [\Pr(X_2^{n-1} = x_2^{n-1} | Y_1^n = \mathbf{y}, X_1 = x_1, X_n = x_n) > 0] \\ &\stackrel{(a)}{=} \sum_{\substack{x_1 \in g^{-1}[y_1] \\ x_n \in g^{-1}[y_n]}} [\Pr(X_1 = x_1, X_n = x_n | Y_1^n = \mathbf{y}) > 0] \end{aligned}$$

$$\leq N^2 < \infty$$

where (a) is due to (11). Since this holds for all n and all realizations, this proves

$$\bar{H}(\mathbf{X}|\mathbf{Y}) = 0 \Rightarrow \exists C < \infty: \sup_{n \rightarrow \infty} T_n \leq C \quad (12)$$

With (3a), the reverse implication is trivial. ■

B. Proof of Corollary 1

The proof employs elementary results from graph theory: Let \mathbf{A} denote the adjacency matrix of the Markov chain, i.e., $A_{i,j} = [P_{i,j} > 0]$. The number of closed walks of length k on the graph determined by \mathbf{A} is given as [15, p. 24]

$$\sum_{i=1}^N \lambda_i^k \quad (13)$$

where $\{\lambda_i\}_{i=1}^N$ is the set of eigenvalues of \mathbf{A} .

Let t_X^k denote the number of sequences $\mathbf{x} \in \mathcal{X}^k$ of \mathbf{X} with positive probability, i.e.,

$$t_X^k = \sum_{\mathbf{x} \in \mathcal{X}^k} [\Pr(X_1^n = \mathbf{x}) > 0] . \quad (14)$$

Clearly, $t_X^k \geq \sum_{i=1}^N \lambda_i^k$. Furthermore, defining t_Y^k similarly we obtain $t_Y^k \leq M^k$. With λ_{\max} denoting the largest eigenvalue of \mathbf{A} ,

$$\frac{t_X^k}{t_Y^k} \geq \frac{\sum_{i=1}^N \lambda_i^k}{M^k} \geq \left(\frac{\lambda_{\max}}{M} \right)^k . \quad (15)$$

If $\lambda_{\max} > M$, then the ratio of possible length- k sequences of \mathbf{X} to those of \mathbf{Y} increases exponentially. Then the *pigeon-hole-principle* implies that also the preimage count T_n is unbounded. Thus,

$$\bar{H}(\mathbf{X}|\mathbf{Y}) = 0 \Rightarrow M \geq \lambda_{\max} . \quad (16)$$

Finally, the *Perron-Frobenius theorem* for non-negative matrices [16, Cor. 8.3.3] bounds the largest eigenvalue of \mathbf{A} from below by the minimum out-degree of \mathbf{P} . ■

C. Proof of Proposition 2

We prove the proposition by contradiction: Assume (\mathbf{P}, h) violates SFS(k). Then there exists a $\mathbf{z} \in \mathcal{Z}^{k-1}$, $z \in \mathcal{Z}$ such that there exist two distinct $\mathbf{x}', \mathbf{x}'' \in h^{-1}[\mathbf{z}]$ and two, not necessarily distinct $x', x'' \in h^{-1}[z]$ such that

$$\Pr(X_2^k = \mathbf{x}' | Z_2^n = \mathbf{z}, X_1 = x') > 0 \quad (17)$$

and

$$\Pr(X_2^k = \mathbf{x}'' | Z_2^n = \mathbf{z}, X_1 = x'') > 0 . \quad (18)$$

In other words, there are two different sequences $\mathbf{x}', \mathbf{x}''$ accessible from either the same ($x' = x''$) or from different ($x' \neq x''$) starting states.

Now take $\mathbf{y} = f(\mathbf{z})$ and $y = f(z)$. Since h is a refinement of g , we have $h^{-1}[\mathbf{z}] \subseteq g^{-1}[\mathbf{y}]$ and $h^{-1}[z] \subseteq g^{-1}[y]$. As a consequence, $\mathbf{x}', \mathbf{x}'' \in g^{-1}[\mathbf{y}]$ and $x', x'' \in g^{-1}[y]$, implying that (\mathbf{P}, g) violates SFS(k). This proves

$$(\mathbf{P}, h) \text{ violates SFS}(k) \Rightarrow (\mathbf{P}, g) \text{ violates SFS}(k) . \quad (19)$$

The negation of these statements completes the proof. ■

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